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The Low Energy, Local Density of States of an Isolated Vortex in an Extreme Type II Superconductor

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The local density of states of low energy electronic excitations in a vortex is investigated. Starting from the Bogoliubov Hamiltonian, we make the approximation of replacing the quadratic momentum dependence of the free particle kinetic energy by a linear dependence. We then work with a linearized order parameter profile of the vortex $\Delta(r)$; $\Delta(r) \approx \Delta'(0) r$, where r is the distance from the vortex axis. The resulting theory has a simple operator structure and is exactly diagonalizable. It leads to a sequence of "oscillator-like" excitations in the core of the vortex. The low energy behaviour is contained in the zeroth oscillator level and when the excitation energy E and vortex profile obey $\Delta(r)^2 - E^2 \ll v_F \Delta'(0)$, where v_F is the Fermi velocity, we find that the leading behaviour in the local density of states is of the form $(\Delta(r)^2 - E^2)^{1/2} \Theta(\Delta(r)^2 - E^2)$ with $\Theta(x)$ a Heaviside step function. This "edge" behaviour is clearly discernible in numerical calculations of the local density of states. © 1993 Academic Press, Inc.

1. INTRODUCTION

Type II superconductors have the ability to support topological excitations known as vortex lines [1]. These are line defects within the superconductor in which the order parameter falls to zero along a line (the vortex axis) and a quantized phase of $2n\pi$ (n = integer) is aquired when the vortex axis is encircled.

The vortex has a significant influence on the spectrum of the electrons that move in its presence. It has the ability to both scatter and bind electronic states [2] and generally causes a significant modification in the local density of electronic states.

The present work is concerned with the local density of states associated with a single straight vortex in a clean, extreme type II superconductor. The "classic" work on the density of states associated with a vortex was carried out by Caroli *et al.* [2]. These authors determined the spatial integral of the local density of states—a global quantity that contains no information on the spatial distribution of the excitations. This quantity, determined by a careful matching of excitation eigenfunctions, was found to be energy independent in the low energy regime $E^2 \ll \Delta_{BCS}^2$, where E is the energy of the excitations relative to the Fermi energy and Δ_{BCS} is the bulk equilibrium gap amplitude. Subsequent work by Bardeen *et al.* [3], using a WKB approach, determined a number of quantities including the spectrum of

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excitations in the presence of a vortex over a wider range of energies than that found in Ref. [2]. More recent work [4–7], has concentrated on the local density of states, a quantity which, when convoluted with the derivative of the Fermi–Dirac distribution, is measurable in conductance experiments using a scanning tunnelling microscope [8].

In Refs. [5–7] the local density of states was calculated by numerically solving Eilenberger's quasiclassical equations for the partially integrated Greens functions [9] and their results display a rich structure in the local density of states. The present work has the aim of analytically calculating the local density of states in a simplified model of the vortex, thereby providing intuition on the numerically determined behaviour given in Refs. [5–7]. The approach presented is limited to low energies and leads to an approximation for the local density of states in this region. A relatively simple functional form for the local density of states is found.

It should be noted that certain aspects of the present work are implicit in the analytical work of Ref. [4], developed further in the Appendix of Ref. [5], where the very different language of quasiclassical theory is used to describe the excitation spectrum.

This paper is arranged as follows. In Section 2 of this work we define the problem under consideration while in Section 3 we show how to approximate the quadratic momentum dependence of the free particle kinetic energy by a linear dependence. In Section 4 an approximate order parameter profile for the vortex is introduced and the resulting theory is exactly diagonalized. Sections 5 and 6 derive the local density of states and investigate some of its properties. Section 7 consists of a discussion and there are five appendices.

We work in units in which $\hbar = 1$, the electronic charge is denoted by $-e$ and a prime, $'$, denotes differentiation of a function with respect to its argument.

2. Definition of the Problem

Extreme type II superconductors are characterised by having a BCS coherence length, ξ_{BCS}, that is small compared with the London magnetic penetration depth, λ_L. In the present work we consider an extreme type II superconductor which contains an isolated vortex lying along the z axis.

The position of quantities, relative to the vortex, in the x, y plane is given in terms of cartesian or cylindrical polar coordinates

$$\mathbf{r} = (x, y) = (r \cos \psi, r \sin \psi). \tag{2.1}$$

For definiteness, we shall use $\mathbf{r}$ throughout the paper to denote such a two-dimensional vector in the x, y plane and likewise, we shall use $\mathbf{x}$ to denote the three-dimensional position vector (x, y, z).

We calculate the low energy, local density of states of the vortex within the framework of a model mean field Hamiltonian. In terms of the Pauli spin matrices

$(\sigma_1, \sigma_2, \sigma_3)$ that here act in the space of spin up articles and spin down holes, the first quantised Bogoliubov Hamiltonian that describes the electronic excitations in a superconductor can be written as[1] [1]

$$H = K(\mathbf{p} + e\mathbf{A}\sigma_3)\,\sigma_3 + V(\mathbf{r}). \tag{2.2a}$$

In this equation K is the kinetic energy measured relative to the Fermi energy $E_F = mv_F^2/2 \equiv k_F^2/2m$:

$$K(\mathbf{p}) = \mathbf{p}^2/2m - k_F^2/2m. \tag{2.2b}$$

Furthermore, $V(\mathbf{r})$ is the order parameter describing the vortex and we shall assume its amplitude, $\Delta(r)$, is real and positive:[2]

$$V(\mathbf{r}) = \Delta(r)\,\sigma_1 \exp[i\psi\sigma_3], \tag{2.2c}$$

with r and ψ taken from Eq. (2.1).

The vector potential $\mathbf{A}$ will be neglected in what follows. It only plays a significant role over distances comparable with the London magnetic penetration length λ_L and we are interested in low energy properties associated with states bound in the radial direction from the vortex. These typically vary on a scale no greater than the BCS coherence length ξ_{BCS} and this is much smaller than λ_L for the extreme type II superconductors under consideration. As a consequence the vector potential leads to a very small contribution in Eq. (2.2a) [2] (in Ref. [10] the leading corrections arising from the vector potential have been considered). Should we have recourse to consider effects associated with scattering states, which are not localised around the vortex, then it would be necessary to keep the vector potential in the Hamiltonian; we shall briefly discuss this point at a later stage.

Including a factor of two, to account for the contribution of both spin projections, the local density of states can be written as[3]

$$\rho(E; \mathbf{x}) = \frac{2}{\pi} \operatorname{Im} G(\mathbf{x}, \mathbf{x}; E - i0_+)_{11}, \tag{2.3}$$

[1] This Hamiltonian omits a paramagnetic contribution following from the interaction of the electronic spins with the magnetic field of the vortex. In extreme type II systems this energy is a small quantity. In superconductors that are not extreme type II in character, Hansen [10] has estimated the effects on the spectrum from including both the paramagnetic term and the vector potential in the Hamiltonian.

[2] The order parameter profile for a vortex, $\Delta(r)$, vanishes at $r = 0$ and achieves a magnitude equal to the bulk value of an equilibrium superconductor at large distances from the vortex: $\lim_{r \to \infty} |\Delta(r)| = \Delta_{BCS}$. In the present work we have selected, for convenience, a profile $\Delta(r)$ that is real and positive. It is possible to rework our results for a general choice of phase of $\Delta(r)$ without changing any observable results which depend on $|\Delta|^2$.

[3] The Bogoliubov Hamiltonian, Eq. (2.2a), describes the energy of excitations relative to the Fermi energy, E_F. Accordingly, the quantity $\rho(E; \mathbf{x})$ of Eq. (2.3) describes the local density of states of excitations having an energy E relative to E_F.

where Im denotes the imaginary part of a quantity, the subscript 11 denotes the (1, 1) element in particle hole space (a space spanned by the Pauli matrices), and $G(\mathbf{x}, \mathbf{x}; E - i0_+)$ is a particular matrix element of the one-particle Greens function

$$G(\mathbf{x}, \mathbf{x}'; \omega) = \langle \mathbf{x} | \frac{1}{\omega - H} | \mathbf{x}' \rangle; \qquad \omega \text{ complex.} \tag{2.4}$$

In this last equation we have suppressed matrix indices and have employed Dirac notation, thus the momentum and coordinate operators appearing in Eq. (2.2a) are to be interpreted as operators obeying $[x_a, p_b] = i\delta_{ab}$ and act in the spaces of bra and ket vectors.

3. Linearization of the Kinetic Energy

In Ref. [11] the free energy associated with a vortex in a superconductor was investigated. In the course of that work the quadratic momentum dependence of the free particle kinetic energy was approximated by a linear momentum dependence. This was a rather accurate approximation that simplified the ensuing calculations and we shall make the same linearization here. There are some differences in the procedure leading to the approximation in the present work compared with the procedure used in Ref. [11] and we present, for completeness, the full details of the linearization here. We shall perform the kinetic energy linearization on $G(\mathbf{x}, \mathbf{x}; \omega)$, since $\rho(E; \mathbf{x})$ is expressed in terms of this function by virtue of Eq. (2.3).

We begin by noting that for $f(\mathbf{p}, \mathbf{x})$ a function of momentum and coordinate operators appropriate to a particle moving in d spatial dimensions we have the following result, which is proven in Appendix A,

$$\langle \mathbf{x} | f(\mathbf{p}, \mathbf{x}) | \mathbf{x} \rangle = \int \frac{d^d k}{(2\pi)^d} f(\mathbf{k} - i\nabla, \mathbf{x}), \tag{3.1}$$

where ∇ acts on quantities to its right. We can use this equation for $d = 3$ and obtain

$$\begin{aligned} G(\mathbf{x}, \mathbf{x}; \omega) &= \langle \mathbf{x} | \frac{1}{\omega - [K(\mathbf{p})\sigma_3 + V(\mathbf{r})]} | \mathbf{x} \rangle \\ &= \int \frac{d^3 k}{(2\pi)^3} \frac{1}{\omega - [K(\mathbf{k} - i\nabla)\sigma_3 + V(\mathbf{r})]}. \end{aligned} \tag{3.2}$$

We then make the approximations

$$\begin{aligned} &K(\mathbf{k} - i\nabla) \approx \varepsilon - iv_F \hat{k} \cdot \nabla, && \text{with} \quad \varepsilon = k^2/2m - k_F^2/2m. \\ &\int \frac{d^3 k}{(2\pi)^3} \approx N(0) \int_{-\infty}^{\infty} d\varepsilon \int \frac{d\Omega_k}{4\pi}, && \text{with} \quad N(0) = \frac{mk_F}{2\pi^2}. \end{aligned} \tag{3.3}$$

$N(0)$ is the free particle density of states/unit volume for a single spin projection and $d\Omega_k$ is the solid angle measure about the unit vector $\hat{k}$. The above approximations are based on the assumption that the typical spatial scale of variation of the order parameter is large compared with k_F^{-1} and that the variation of the free particle density of states at the Fermi energy may be neglected. From Eqs. (3.2) and (3.3) it follows that

$$G(\mathbf{x}, \mathbf{x}; \omega) \approx N(0) \int_{-\infty}^{\infty} d\varepsilon \int \frac{d\Omega_k}{4\pi} \frac{1}{\omega - [(\varepsilon - iv_F \hat{k} \cdot \nabla)\sigma_3 + V(\mathbf{r})]}. \tag{3.4}$$

Next we parametrize $\hat{k}$ by polar and azimuthal angles ϑ and φ and introduce the triad of orthonormal vectors $(\hat{z}, \hat{e}_1, \hat{e}_2)$:

$$\hat{k} = \cos\vartheta\, \hat{z} + \sin\vartheta\, \hat{e}_1 \tag{3.5a}$$

$$\hat{z} = (0, 0, 1)$$

$$\hat{e}_1 = (\cos\varphi, \sin\varphi, 0) \tag{3.5b}$$

$$\hat{e}_2 = (-\sin\varphi, \cos\varphi, 0).$$

We furthermore introduce coordinates (ζ, η),

$$\mathbf{r} = \zeta \hat{e}_1 + \eta \hat{e}_2, \tag{3.6}$$

and it follows, since there is no z dependence in the order parameter $V(\mathbf{r})$, that

$$\hat{k} \cdot \nabla \equiv \sin\vartheta\, \hat{e}_1 \cdot \nabla = \sin\vartheta\, \partial_\zeta \qquad (\partial_\zeta \equiv \partial/\partial\zeta). \tag{3.7}$$

The change of variable from ε to k' (a variable which may be positive or negative) defined by

$$\varepsilon = v_F k' \sin\vartheta. \tag{3.8}$$

is then made and Eqs. (3.4) to (3.8) allow us to write (we neglect the prime on k)

$$G(\mathbf{x}, \mathbf{x}; \omega) \approx 2\pi v_F N(0) \int \frac{d\Omega_k}{4\pi} \sin\vartheta \int \frac{dk}{2\pi} \frac{1}{\omega - [v_F \sin\vartheta(k - i\partial_\zeta)\sigma_3 + V(\mathbf{r})]}. \tag{3.9}$$

The last step in the linearisation of the kinetic energy is to observe that the k integral appearing in Eq. (3.9) is of exactly the form appearing in the right-hand side of Eq. (3.1) with $d = 1$, allowing us to use that equation from right to left and to obtain the $\langle\zeta| \cdots |\zeta\rangle$ matrix element of a "one-dimensional" Greens function that depends on the coordinate operator $\zeta \equiv \hat{e}_1 \cdot \mathbf{x}$ and its conjugate momentum[4]

$$p_1 = \hat{e}_1 \cdot \mathbf{p} \tag{3.10a}$$

$$G(\mathbf{x}, \mathbf{x}; \nu) \approx 2\pi v_F N(0) \int \frac{d\Omega_k}{4\pi} \sin\vartheta \langle\zeta| \frac{1}{\omega - [v_F \sin\vartheta\, p_1 \sigma_3 + V(\mathbf{r})]} |\zeta\rangle. \tag{3.10b}$$

[4] It should be noted that $p_1 \equiv \hat{e}_1 \cdot \mathbf{p}$ is a linear combination of p_x and p_y. Since ϑ and φ are independent of x and y it follows that p_1 commutes with functions of ϑ and φ.

We must be bear in mind that, since we use the coordinates (ζ, η), we must express $V(\mathbf{r})$ in terms of these. A simple calculation yields

$$\zeta = r\cos(\psi - \varphi), \qquad \eta = r\sin(\psi - \varphi) \tag{3.11}$$

$$V(\mathbf{r}) = \frac{\Delta(r)}{r}\sigma_1 \exp[i\varphi\sigma_3](\zeta + i\sigma_3\eta). \tag{3.12}$$

Furthermore, by virtue of Eq. (3.11), both ζ and η depend on the azimuthal angle of $\hat{k}$, namely φ, and this appears in the solid angle Ω_k. Thus after evaluating the matrix element in Eq. (3.10b) we must set ζ and η to their values given in Eq. (3.11) and only then perform the Ω_k integration.

Purely as a matter of convenience, it is possible to remove the factor $\exp[i\varphi\sigma_3]$ appearing in the order parameter of Eq. (3.12) by performing a unitary transformation. We have

$$v_F \sin\vartheta\, p_1\sigma_3 + V(\mathbf{r}) = U[v_F \sin\vartheta\, p_1\sigma_3 + \tilde{V}(\mathbf{r})]\, U^+ \tag{3.13a}$$

$$U = \exp[-i\varphi\sigma_3/2] \tag{3.13b}$$

$$\tilde{V}(\mathbf{r}) = \frac{\Delta(r)}{r}(\sigma_1\zeta + \sigma_2\eta); \tag{3.13c}$$

thus the Greens function incorporating the linearized kinetic energy and the unitary transformation, Eq. (3.13b), is

$$G(\mathbf{x}, \mathbf{x}; \omega) \approx 2\pi v_F N(0) \int \frac{d\Omega_k}{4\pi} \times \sin\vartheta \left[\left\{ U \langle\zeta| \frac{1}{\omega - [v_F \sin\vartheta\, p_1\sigma_3 + \tilde{V}(\mathbf{r})]} |\zeta\rangle\, U^+ \right\}\Big|_{\substack{\zeta = r\cos(\psi \quad \varphi) \\ \eta = r\sin(\psi \quad \varphi)}} \right]. \tag{3.14}$$

4. Linearization of the Order Parameter Profile and Diagonalization of the Resulting Hamiltonian

The low energy electronic excitations of the system under consideration are states which are bound radially in the core of the vortex but which are able to translate freely along the axis of the vortex. The lowest lying of these are expected to have eigenfunctions which decay to zero, in the radial direction, after a small distance from the vortex axis. They are therefore sensitive to details of the order parameter in the vicinity of $r = 0$. As an estimate of the low energy local density of states, we shall approximate the exact order parameter profile, $\Delta(r)$, by the lowest non-trivial

term in its Taylor series expansion about $r = 0$. Since $\Delta(0)$ vanishes the approximation amounts to a linearization:

$$\Delta(r) \approx \Delta_{\mathrm{L}}(r) \equiv \Delta'(0)\, r. \tag{4.1}$$

The Hamiltonian that incorporates the above linearized order parameter profile is denoted by h and is given by

$$\begin{aligned} h &= v_{\mathrm{F}} \sin\vartheta\, p_1 \sigma_3 + \frac{\Delta_{\mathrm{L}}(r)}{r}(\sigma_1 \zeta + \sigma_2 \eta) \\ &\equiv v_{\mathrm{F}} \sin\vartheta\, p_1 \sigma_3 + \Delta'(0)(\sigma_1 \zeta + \sigma_2 \eta) \end{aligned} \tag{4.2}$$

and the resulting approximation to the Greens function, obtained by using this Hamiltonian, is denoted by G_{L}:

$$G_{\mathrm{L}}(\mathbf{x}, \mathbf{x}; \omega) \equiv 2\pi v_{\mathrm{F}} N(0) \int \frac{d\Omega_k}{4\pi} \sin\vartheta \left[U \langle\zeta| \frac{1}{\omega - h} |\zeta\rangle\, U^{+} \Big|_{\substack{\zeta = r\cos(\psi - \varphi) \\ \eta = r\sin(\psi - \varphi)}} \right]. \tag{4.3}$$

If we square h we obtain, amongst other terms, a harmonic oscillator Hamiltonian for the conjugate dynamical variables ζ and p_1 (η is a c-number)

$$h^2 = p_1^2/2\mu + \mu\Omega^2(\zeta^2 + \eta^2)/2 + (\Omega/2)\,\sigma_2, \tag{4.4a}$$

where the mass μ and frequency Ω are given by[5]

$$\mu = \frac{1}{2(v_{\mathrm{F}} \sin\vartheta)^2}, \qquad \Omega = 2v_{\mathrm{F}}\, \Delta'(0) \sin\vartheta. \tag{4.4b}$$

Equation (4.4a), which relates h to the Hamiltonian of a harmonic oscillator, makes it natural to express h in terms of the usual annihilation and creation operators a and a^{+} of the harmonic oscillator:

$$a = \sqrt{\mu\Omega/2}\, \zeta + ip_1/\sqrt{2\mu\Omega}, \qquad a^{+} = \sqrt{\mu\Omega/2}\, \zeta - ip_1/\sqrt{2\mu\Omega}. \tag{4.5}$$

Furthermore, we also use

$$\sigma^{\pm} = (\sigma_3 \pm i\sigma_1)/2. \tag{4.6}$$

[5] There should be no confusion between the oscillator frequency Ω and the solid angle Ω_k, the latter only appearing in the measure of angular integrals. Note that we have assumed $\Delta'(0)$ is positive. This is compatible with the choice of order parameter phase specified in Section 2. Furthermore, since ϑ lies in the range 0 to π, $\sin\vartheta$ is positive and consequently the oscillator frequency Ω is positive.

These are the stepping operators for σ_2; i.e., if $|\sigma\rangle$ is an eigenvector of σ_2 with eigenvalue σ $(=\pm 1)$ then

$$\begin{aligned} &\sigma_2 |\sigma\rangle = \sigma |\sigma\rangle, && \sigma = \pm 1 \\ &\sigma^+ |-1\rangle = |+1\rangle, && \sigma^+ |+1\rangle = 0, \\ &\sigma^- |-1\rangle = 0, && \sigma^- |+1\rangle = |-1\rangle. \end{aligned} \tag{4.7}$$

In terms of the above operators we find

$$h = \sqrt{\Omega}\, i(\sigma^- a^+ - \sigma^+ a) + \sqrt{\mu\Omega^2/2}\, \sigma_2 \eta. \tag{4.8}$$

We need to determine both the eigenvectors and eigenvalues of h in order to calculate G_L and to this purpose we use the basis vectors $|n, \sigma\rangle$, where

$$\begin{aligned} &a^+ a |n, \sigma\rangle = n |n, \sigma\rangle, && n = 0, 1, 2, 3, ..., \\ &\sigma_2 |n, \sigma\rangle = \sigma |n, \sigma\rangle, && \sigma = \pm 1. \end{aligned} \tag{4.9}$$

A little consideration indicates that eigenvectors of h can be written as a linear combination of $|n, -1\rangle$ and $|n-1, +1\rangle$. In terms of the parameter

$$\lambda \equiv \sqrt{\mu\Omega^2/2}\, \eta, \tag{4.10}$$

it may be verified that for $n \neq 0$,

$$\left.\begin{aligned} &\text{Eigenvalues} = E_{ns} = s\sqrt{\lambda^2 + n\Omega}, \\ &\text{Eigenvectors} = |E_{ns}\rangle = u_{ns} |n, -1\rangle + v_{ns} |n-1, +1\rangle, \end{aligned}\right\} \quad \begin{aligned} &s = \pm 1 \qquad (4.11a) \\ &n = 1, 2, 3, ..., \qquad (4.11b) \end{aligned}$$

$$u_{ns} = \sqrt{\tfrac{1}{2}(1 - \lambda/E_{ns})}, \qquad v_{ns} = -is\sqrt{\tfrac{1}{2}(1 + \lambda/E_{ns})}, \tag{4.11c}$$

while for $n = 0$ the eigenvector is unpaired; the eigenvalue is not accompanied by one of the opposite sign:

$$\left.\begin{aligned} &\text{Eigenvalue} = E_0 = -\lambda, \\ &\text{Eigenvector} = |E_0\rangle = |0, -1\rangle, \end{aligned}\right\} \quad n = 0. \qquad \begin{aligned} &(4.12a) \\ &(4.12b) \end{aligned}$$

The diagonalization of the Hamiltonian h containing the linearized vortex profile has led to a sequence of "oscillator levels" in the core of the vortex. We shall use these to calculate the local density of states.

5. Determination of the Local Density of States for the Linearised Vortex Profile

For the calculation of the local density of states, we introduce a resolution of unity into Eq. (4.3) for G_L. It takes the form

$$1 = |E_0\rangle\langle E_0| + \sum_s \sum_{n=1}^{\infty} |E_{ns}\rangle\langle E_{ns}| \tag{5.1}$$

and we obtain

$$G_L(\mathbf{x}, \mathbf{x}; \omega) = 2\pi v_F N(0) \int \frac{d\Omega_k}{4\pi} \sin\vartheta \left[U \left\{ \frac{\langle \zeta | E_0 \rangle \langle E_0 | \zeta \rangle}{\omega - E_0} + \sum_{s} \sum_{n=1}^{\infty} \frac{\langle \zeta | E_{ns} \rangle \langle E_{ns} | \zeta \rangle}{\omega - E_{ns}} \right\} U^+ \right] \Bigg|_{\substack{\zeta = r\cos(\psi - \varphi) \\ \eta = r\sin(\psi - \varphi)}} . \tag{5.2}$$

Next we write the energy eigenfunctions of the harmonic oscillator (whose mass and frequency are given in Eq. (4.4b)) as

$$\langle \zeta | n \rangle = \Phi_n(\zeta). \tag{5.3}$$

We furthermore use the Dirac identity

$$\frac{1}{\pi} \operatorname{Im} \frac{1}{E - x - i0_+} = \delta(E - x) \tag{5.4}$$

and, after a straightforward calculation, obtain an approximation for the local density of states following from the linearised vortex profile:

$$\rho(E; \mathbf{x}) \approx \rho_L(E; \mathbf{x}) \equiv \frac{2}{\pi} \operatorname{Im} G_L(\mathbf{x}, \mathbf{x}; E - i0_+)_{11}$$

$$= 4\pi v_F N(0) \int \frac{d\Omega_k}{4\pi} \sin\vartheta \frac{1}{2} \left[\left\{ \Phi_0(\zeta)^2 \, \delta(E - E_0) + \sum_{s} \sum_{n=1}^{\infty} [|u_{ns}|^2 \, \Phi_n(\zeta)^2 + |v_{ns}|^2 \, \Phi_{n-1}(\zeta)^2] \, \delta(E - E_{ns}) \right\} \Bigg|_{\substack{\zeta = r\cos(\psi - \varphi) \\ \eta = r\sin(\psi - \varphi)}} \right]. \tag{5.5}$$

There are two simplifications of this formula that can be made.

First, the azimuthal angle φ, which is integrated over a range of 2π, can be shifted by the "external" angle ψ (see Eq. (2.1)); $\varphi \to \varphi + \psi$. This makes it manifest that the local density of states depends, as a function of position, only on r, the radial distance from the vortex.

Second, the quantities $|u_{ns}|^2$ and $|v_{ns}|^2$ that appear in the above sum may both, as a result of the φ integration, be effectively replaced by $\frac{1}{2}$. This can be seen to come about by considering what happens when φ is replaced by $\varphi + \pi$. Under this change η changes sign and hence, via Eq. (4.10), so do the λ's that appear in the coefficients u_{ns} and v_{ns} given in Eq. (4.11c). Thus on combining the contributions of φ and $\varphi + \pi$, the λ's within $|u_{ns}|^2$ and $|v_{ns}|^2$ cancel to zero on φ integrating.

The above simplifications allow us to write

$$\rho_L(E;\mathbf{x}) = 2\pi v_F N(0) \int \frac{d\Omega_k}{4\pi} \sin\vartheta \left[\left\{ \Phi_0(\zeta)^2\, \delta(E-E_0) + \sum_s \sum_{n=1}^{\infty} \frac{1}{2} \left[\Phi_n(\zeta)^2 + \Phi_{n-1}(\zeta)^2 \right] \delta(E-E_{ns}) \right\} \Bigg|_{\substack{\zeta = r\cos(\varphi) \\ \eta = -r\sin(\varphi)}} \right]. \tag{5.6}$$

This formula still requires the evaluation of the ϑ and φ integrals for its explicit determination. As we shall show below, the dominant contribution to the local density of states at energies $E^2 \ll 2v_F \Delta'(0)$ follows from the single unpaired level with $n=0$, the zeroth oscillator level. Denoting its contribution to the local density of states by $\rho_L(E;\mathbf{x})^{(0)}$ we have, on substituting the known form for the ground-state of the harmonic oscillator of mass and frequency given by Eq. (4.4b):

$$\begin{aligned} \rho_L(E;\mathbf{x})^{(0)} &= 2\pi v_F N(0) \int \frac{d\Omega_k}{4\pi} \sin\vartheta\, \Phi_0(\zeta)^2\, \delta(E-E_0) \Big|_{\substack{\zeta = r\cos(\varphi) \\ \eta = -r\sin(\varphi)}} \\ &= N(0) \sqrt{v_F \Delta'(0)/\pi} \int_0^{2\pi} d\varphi \int_0^{\pi/2} d\vartheta \sin^{3/2}\vartheta \exp\left[\frac{-r^2 \Delta'(0) \cos^2\varphi}{v_F \sin\vartheta} \right] \\ &\quad \times \delta(E - \Delta'(0)\, r \sin\varphi). \end{aligned} \tag{5.7}$$

The delta function makes the φ integration straightforward (two values of φ contribute) and leads to

$$\begin{aligned} \rho_L(E;\mathbf{x})^{(0)} &= \frac{2}{\sqrt{\pi}} N(0) \sqrt{v_F \Delta'(0)} \frac{\Theta(\Delta_L(r)^2 - E^2)}{\sqrt{\Delta_L(r)^2 - E^2}} \\ &\quad \times \int_0^{\pi/2} d\vartheta \sin^{3/2}\vartheta \exp\left[-\frac{(\Delta_L(r)^2 - E^2)}{v_F \Delta'(0) \sin\vartheta} \right], \end{aligned} \tag{5.8}$$

where $\Theta(x)$ is the Heaviside step function and all the r dependence has been expressed in terms of the linearised order parameter profile $\Delta_L(r)$ given in Eq. (4.1).

An alternative representation of $\rho_L(E;\mathbf{x})^{(0)}$ which may be useful in some circumstances uses a result involving the Bessel function of imaginary argument K_0 [12] which is proved in Appendix B:

$$\int_0^{\pi/2} d\vartheta \sin^{3/2}\vartheta \exp[-w/\sin\vartheta] = \frac{1}{\Gamma(5/2)} \int_0^{\infty} dv\, v^{3/2} K_0(w+v). \tag{5.9}$$

Using the above result we have

$$\rho_L(E;\mathbf{x})^{(0)} = \frac{8}{3\pi} N(0) \sqrt{v_F \Delta'(0)} \frac{\Theta(\Delta_L(r)^2 - E^2)}{\sqrt{\Delta_L(r)^2 - E^2}} \int_0^{\infty} dv\, v^{3/2} K_0\left(v + \frac{(\Delta_L(r)^2 - E^2)}{v_F \Delta'(0)} \right). \tag{5.10}$$

It is noteworthy that in the above formulae for $\rho_L(E;\mathbf{x})^{(0)}$ the only quantity appearing with the dimensions of (energy)^2, apart from the quantities E^2 and $\Delta_L(r)^2$, is the parameter $v_F\,\Delta'(0)$ and this quantity sets the natural scale of energy of the local density of states. If we write $\Delta'(0)=\Delta_{BCS}/\xi$ with Δ_{BCS} the bulk gap amplitude and ξ the "core size" of the vortex [6, 11, 13] then the natural scale of (energy)^2 in the problem is $\pi(\xi_{BCS}/\xi)\,\Delta_{BCS}^2$, where[6] $\xi_{BCS}=v_F/\pi\,\Delta_{BCS}$.

Equation (5.8) (or Eq. (5.10)) has been derived assuming a linear dependence of the order parameter profile $\Delta(r)$ and this sets the validity as a function of r. When $\Delta(r)$ is appreciably different from its linearized approximation, $\Delta_L(r)$, Eq. (5.8) is no longer a good approximation to the local density of states.

Let us return to our promise, made above, of showing that it is a good approximation to write

$$\rho_L(E;\mathbf{x})\approx\rho_L(E;\mathbf{x})^{(0)} \qquad \text{for} \quad E^2\ll 2v_F\,\Delta'(0). \tag{5.11}$$

In Appendix C, we estimate the contribution of the excited states that appear in the sum over n in Eq. (5.6). A comparison of Eqs. (C.10), (C.11), and Eq. (5.8), in the appropriate limits, indicates that the contribution from the terms with $n\geqslant 1$ is down on the $n=0$ term (when it is non-zero) by a factor $\lesssim(E^2/(2v_F\,\Delta'(0))^{5/2}$.

6. Investigation of the Formula Derived for the Local Density of States

Equation (5.8) or (5.10) for the local density of states is a complicated object and in this section we shall investigate some of its properties.

(i) *Total Density of States/Unit Length at $E=0$*

The total (or global) density of states/unit length of vortex is obtained by integrating the local density of states over the x, y plane. It is defined by

$$\rho_L(E)\equiv\int d^2r\,\rho_L(E;\mathbf{x})=2\pi\int_0^\infty r\,dr\,\rho_L(E;\mathbf{x}), \tag{6.1}$$

the last equality following from the cylindrical symmetry of $\rho_L(E;\mathbf{x})$. Let us consider this quantity for the case of very low energies by setting $E=0$ in Eq. (5.8). To evaluate $\rho_L(0)$ it is simplest to work with the $n=0$ term from Eq. (5.6) specialised to $E=0$; the normalisation integral of Φ_0^2 determines the r integral and an elementary trigonometric integral is left. The result obtained is[7]

$$\rho_L(0)=\frac{\pi^2}{2}\frac{v_F}{\Delta'(0)}N(0). \tag{6.2}$$

[6] If the core size of the vortex, ξ, is so small that $(k_F\xi)^{-1}$ is not a small parameter (cf. Ref. [13]) then the kinetic energy linearization of Section 3 will not be a good approximation to the full quadratic momentum dependence.

[7] No superscript of zero need be attached to the local density of states appearing on the left-hand side of this equation since the contribution from the $n\neq 0$ states is zero for $E=0$; see Eq. (C.10).

This coincides with the result presented by Caroli *et al.* [2] when their order parameter profile is taken to be the linearized one of Eq. (4.1). There is one point of discrepancy, however. The result of Caroli *et al.* is stated [2] to include only a single spin projection, whereas the result of this work included both spin projections. There appears, therefore, to be a factor of two discrepancy between the two results. We have carried out tests on the approach used in the present work with no discrepancies appearing and we maintain that Eq. (6.2) is the low energy density of states including *both* spin projections. More discussion of this point is contained in Appendix D.

(ii) *Behaviour of the Local Density of States as a Function of Energy at Fixed r*

Equation (5.8) or (5.10) for the approximate local density of states has, as a function of energy at fixed r, a striking feature in the form of the singular factor $(\Delta_L(r)^2 - E^2)^{-1/2}\,\Theta(\Delta_L(r)^2 - E^2)$. The integral in, e.g., Eq. (5.10) is smooth and when $\Delta_L(r)^2 - E^2 \ll v_F\,\Delta'(0)$ the integral can be replaced by [12]

$$\int_0^\infty dv\, v^{3/2} K_0(v) = 2^{1/2}[\Gamma(5/4)]^2, \tag{6.3}$$

leading to the approximation

$$\rho_L(E;\mathbf{x})^{(0)} \approx \frac{[\Gamma(1/4)]^2}{6\pi} N(0) \sqrt{2v_F\,\Delta'(0)}\, \frac{\Theta(\Delta_L(r)^2 - E^2)}{\sqrt{\Delta_L(r)^2 - E^2}}, \qquad \Delta_L(r)^2 - E^2 \ll v_F\,\Delta'(0). \tag{6.4}$$

This is a remarkably simple result,[8] indicating a very sharp structure occurring at $E = \Delta_L(r)$. We present a plot of this function as a function of E for a fixed value of r. There is strong evidence for this sort of behaviour in the numerical findings of other authors. For example, in Fig. 2e of Ref. [7] the leading peak in the local density of states has an "edge" like structure which lies very close[9] to $E = \Delta_L(r)$. Additionally, in Fig. 2 of Ref. [6] there is a sharp peak in the density of states; however, it occurs at $E \approx \frac{2}{3}\Delta_L(r)$: we believe that in this reference the axis has been mislabelled and requires a rescaling by a factor $\approx \frac{3}{2}$.

A further feature of Eq. (6.4) is the value of the local density of states it leads to at $E = 0$:

$$\rho_L(0;\mathbf{x}) \approx \frac{[\Gamma(1/4)]^2}{6\pi} \frac{N(0)\sqrt{2v_F\,\Delta'(0)}}{\Delta_L(r)}, \qquad \Delta_L(r)^2 \ll v_F\,\Delta'(0). \tag{6.5}$$

[8] It is tempting, and probably reasonable, to incorporate some of the effects of non-linearity of the order parameter profile, $\Delta(r)$, into the above formula, Eq. (6.4), by simply replacing $\Delta_L(r)$ by $\Delta(r)$.

[9] The fact that Ref. [7] used a cylindrical Fermi surface and not a spherical one only shows up in Eq. (6.4) as an overall numerical factor following from a modification of the ϑ integrals. Additional features are present in Fig. 2 of Ref. [7] compared with the simple model of Eq. (6.4). The most obvious is the additional peak at the edge of the continuum $E \sim \Delta_{BCS}$. It seems plausible that in the hyperbolic tangent order parameter profile of Ref. [7] a second level (in the sense of the present work) lies on the edge of the continuum and makes its contribution to the local density of states.

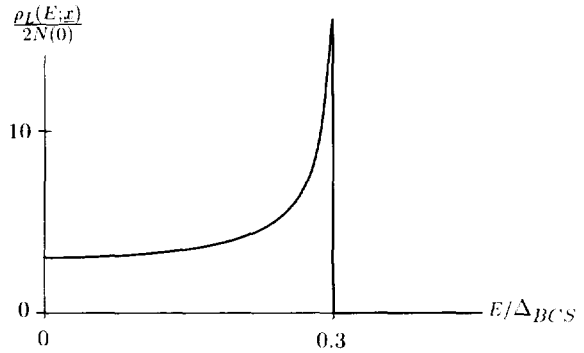

FIG. 1. The approximate density of states $\rho_L(E, \mathbf{x})^{(0)}$, given in Eq. (6.4), is plotted in units of the normal state result $2N(0)$ as a function of E with $r = |\mathbf{x}|$ held fixed. Following Ref. [6] we have taken the order parameter amplitude to be $\Delta_L(r) = \Delta_{BCS} r/\xi_{BCS}$ with $\xi_{BCS} = v_F/\pi\,\Delta_{BCS}$ and Δ_{BCS} the bulk gap amplitude. The value of r selected is $r = 0.3\xi_{BCS}$ such that $\Delta_L(r) = 0.3\,\Delta_{BCS}$. The sharp "edge" structure discernible in the figure occurs at $E = \Delta_L(r) = 0.3\,\Delta_{BCS}$.

This should be directly comparable, in the small r region, with results presented in Fig. 2 of Ref. [6]. To make the comparison with this reference, we note that the author takes, for $r < \xi_{BCS}$, the order parameter $\Delta_{BCS} r/\xi_{BCS}$ with $\xi_{BCS} = v_F/\pi\,\Delta_{BCS}$. Normalising with respect to the normal density of states (which is $2N(0)$ in the present work) and using the order parameter of Ref. [6] we obtain[10] $\rho_L(0; \mathbf{x})/2N(0) \approx c(r/\xi_{BCS})^{-1}$ with $c = [\Gamma(1/4)]^2/(6\sqrt{2\pi}) \approx 0.874$. Thus for $r/\xi_{BCS} = 0.3$ we obtain $\rho_L(0; \mathbf{x})/2N(0) \approx 2.91...$, while from Fig. 2 of Ref. [6] we estimate the same quantity to be approximately 2.9, indicating a close agreement.

It should be remarked that Eq. (6.4) has a familiar look to it because a factor similar to its singular denominator appears in the *high energy* local density of states. To zeroth order in derivatives [14] one obtains a *high energy* density of states which is proportional to $|E|\,(E^2 - \Delta(\mathbf{x})^2)^{1/2}\,\Theta(E^2 - \Delta(\mathbf{x})^2)$ and here we see the order of E^2 and Δ^2, in the denominator, has been reversed relative to Eq. (6.4). This similar behaviour seems to stem from the effective one-dimensional character of motion of particles near the Fermi energy, but in all other respects the formulae seem to have quite distinct origins.

7. DISCUSSION

In this work we have obtained an approximation to the low energy local density of states for a vortex. Our approach started with the Bogoliubov Hamiltonian and involved linearizing the momentum dependence of the kinetic energy. As an

[10] It may be noted that Eq. (6.5) may be written in a form that is independent of the choice of length scale used to describe the vortex. It is $\rho_L(0; \mathbf{x})/\rho_L(0; \mathbf{x}') \approx |\mathbf{x}'|/|\mathbf{x}| \equiv r'/r$, $\Delta_L(r)^2 \ll v_F\,\Delta'(0)$.

intermediate step we derived Eq. (3.14) which gave certain elements of the Greens function in terms of the "one-dimensional" operator

$$v_F \sin\vartheta\, p_1\sigma_3 + \tilde{V}(\mathbf{r}) \equiv v_F \sin\vartheta\, p_1\sigma_3 + \frac{\Delta(r)}{r}(\sigma_1\zeta + \sigma_2\eta). \tag{7.1}$$

The Hamiltonian in Eq. (7.1) is a simple operator that depends on the canonically conjugate operators p_1 and ζ, along with the c-number parameters ϑ and φ, and as such we have a two-parameter family of operators labelled by these c-number variables. A complete determination of local spectral properties of the vortex, such as the density of states, requires knowledge of the eigenvalues and eigenvectors of the above operator over the full ranges of ϑ and φ.

In the present work we approximated $\Delta(r)$ and exploited the formal operator content of the theory to calculate to local density of states. It turned out that the eigenstate of Eq. (7.1) belonging to the eigenvalue of smallest magnitude (calculated with the approximate $\Delta(r)$) contained essentially all of the low energy behaviour. The resulting structure in the local density of states had its origins in a weighted average, determined by the angle ϑ, over the contributions from the radially bound aspect of the state along with the free translational motion along the vortex axis.

Despite the essential simplicity of the calculation presented in this work, we cannot help feeling that we have not done Eq. (7.1) full justice. The operator in question contains essentially full information on the excitation states that are either bound or scattered[11] by the vortex. It seems very plausible that there exist approximations or treatments of $\Delta(r)$ different to the one used here, which are able to capture something other than information on the low lying bound states.

Appendix A

Proof of the Relation $\langle \mathbf{x}|\, f(\mathbf{p}, \mathbf{x})\, |\mathbf{x}\rangle = \int (d^dk/(2\pi)^d)\, f(\mathbf{k} - i\nabla, \mathbf{x})$. (This result is used in Section 3.) For a function of momentum and coordinate operators $f(\mathbf{p}, \mathbf{x})$ appropriate to a particle moving in d dimensions we have

$$\begin{aligned}\langle \mathbf{x}'|\, f(\mathbf{p}, \mathbf{x})\, |\mathbf{x}''\rangle &= f(-i\nabla', \mathbf{x}')\, \delta(\mathbf{x}' - \mathbf{x}'') \\ &= f(-i\nabla', \mathbf{x}') \int \frac{d^dk}{(2\pi)^d} \exp[i\mathbf{k}\cdot(\mathbf{x}' - \mathbf{x}'')] \\ &= \int \frac{d^dk}{(2\pi)^d} \exp[i\mathbf{k}\cdot(\mathbf{x}' - \mathbf{x}'')]\, f(\mathbf{k} - i\nabla', \mathbf{x}')\end{aligned} \tag{A.1}$$

[11] For the scattering states it is necessary to include an important contribution of the magnetic field by incorporating the vector potential into the problem in the original form it appeared in Eq. (2.2a), i.e., by making the substitution $\mathbf{p} \to \mathbf{p} + e\mathbf{A}\sigma_3$ in Eq. (7.1). In Appendix E we show, by performing a unitary transformation, that the vector potential effectively cuts off the long distance spatial variation of the phase of the order parameter, thereby allowing the scattering states to propagate freely at sufficiently large distances from the vortex.

with ∇ acting on quantities to its right. We then set $\mathbf{x}' = \mathbf{x}'' = \mathbf{x}$ and obtain the given equation.

APPENDIX B

Proof of $\int_0^{\pi/2} d\vartheta \sin^{3/2}\vartheta \exp[-w/\sin\vartheta] = (1/\Gamma(5/2)) \int_0^\infty dv\, v^{3/2} K_0(w+v)$. (This relation is used in Section 5 of this work.) The ϑ integral is referred to as $I(w)$ and, in terms of the new variable

$$t = 1/\sin\vartheta, \tag{B.1}$$

it takes the form

$$I(w) = \int_1^\infty \frac{dt}{\sqrt{t^2-1}}\, t^{-5/2} e^{-wt}. \tag{B.2}$$

This is compared with the Bessel function of the imaginary argument $K_0(w)$ which can be written [12]

$$K_0(w) = \int_1^\infty \frac{dt}{\sqrt{t^2-1}}\, e^{-wt}. \tag{B.3}$$

We can express Eq. (B.2) in terms of $K_0(w)$ by using the identity

$$t^{-5/2} = \frac{1}{\Gamma(5/2)} \int_0^\infty dv\, v^{3/2} e^{-vt}. \tag{B.4}$$

Thus using Eq. (B.4) in Eq. (B.2) and interchanging orders of integration we obtain

$$I(w) = \frac{1}{\Gamma(5/2)} \int_0^\infty dv\, v^{3/2} \int_1^\infty \frac{dt}{\sqrt{t^2-1}}\, e^{-(w+v)t}; \tag{B.5}$$

and using the Bessel function representation of Eq. (B.3) leads immediately to the given result.

APPENDIX C: ESTIMATION OF THE CONTRIBUTION OF THE STATES WITH $n \geqslant 1$ TO THE LOCAL DENSITY OF STATES

In this appendix we provide an estimate of the contribution of the levels with $n \neq 0$ in Eq. (5.6) to the local density of states. For notational simplicity we write

$$\lambda = \sqrt{\mu\Omega^2/2}\, \eta = -\Delta_L(r) \sin\varphi \tag{C.1}$$

and omit the argument of Δ_L. We begin in Eq. (5.6) by writing

$$\sum_s \delta(E - E_{ns}) = 2\,|E|\,\delta(E^2 - \Delta_L^2 \sin^2\varphi - n\Omega) \tag{C.2}$$

and use this to perform the integration over φ. It quickly leads to

$$\rho_L(E;\mathbf{x})^{(n>0)} = 2v_F N(0)\,|E| \int_0^{\pi/2} d\vartheta \sin^2\vartheta \sum_{n=1}^{\infty} \frac{\Theta(E^2 - n\Omega)}{\sqrt{(E^2 - n\Omega)}} \frac{\Theta(\Delta_L^2 - E^2 + n\Omega)}{\sqrt{(\Delta_L^2 - E^2 + n\Omega)}}$$
$$\times [\Phi_n(r\sqrt{(\Delta_L^2 - E^2 + n\Omega)/\Delta_L^2})^2 + \Phi_{n-1}(r\sqrt{(\Delta_L^2 - E^2 + n\Omega)/\Delta_L^2})^2]. \tag{C.3}$$

The first Θ function determines the upper limit of the ϑ integral,

$$\sin\vartheta < E^2/(2nv_F\,\Delta'(0)) \tag{C.4}$$

and if we assume

$$E^2 \ll 2v_F\,\Delta'(0) \tag{C.5}$$

then Eq. (C.4) implies, for all n, that $\sin\vartheta$ will be small compared with unity and we can replace $\sin\vartheta$ by ϑ itself to a good approximation:

$$\sin\vartheta \approx \vartheta. \tag{C.6}$$

Next we set

$$\Phi_n(\zeta)^2 + \Phi_{n-1}(\zeta)^2 = L^{-1} f_n(\zeta/L) \tag{C.6a}$$
$$L \approx \sqrt{v_F \vartheta/\Delta'(0)} \tag{C.6b}$$

with f_n a dimensionless function of order unity or smaller. On going to the new integration variable

$$t = 2nv_F\,\Delta'(0)\,\vartheta/E^2, \tag{C.7}$$

we obtain

$$\rho_L(E;\mathbf{x})^{(n>0)} = 2N(0)\sqrt{v_F\,\Delta'(0)}\,(E^2/(2v_F\,\Delta'(0))^{5/2} \int_0^1 dt\, t^{3/2}(1-t)^{1/2}$$
$$\times \frac{\Theta(\Delta_L^2 - E^2(1-t^2))}{\sqrt{(\Delta_L^2 - E^2(1-t^2))}} \sum_{n=1}^{\infty} n^{-5/2} f_n(\sqrt{(2n/t)(\Delta_L^2/E^2 - (1-t))}). \tag{C.8}$$

This equation is a convenient place to estimate $\rho_L(E;\mathbf{x})^{(n>0)}$, since much of the E dependence has been separated out. We shall look at Eq. (C.8) in two limits:

(a) $E^2 \ll \Delta_L(r)^2 \equiv \Delta'(0)^2 r^2$. Here the Gaussian present in the f_n's cause the result to be dominated by the term in the sum with $n = 1$ and to be exponentially small:

$$f_n(\sqrt{(2n/t)(\Delta_L^2/E^2)}) \sim \exp[-(2n/t)\,\Delta_L^2/E^2] \tag{C.9}$$

and

$$\rho_L(E;\mathbf{x})^{(n>0)} \sim N(0)\sqrt{v_F\,\Delta'(0)}\,(E^2/(2v_F\,\Delta'(0))^{5/2}\exp[-2\Delta_L^2/E^2]. \tag{C.10}$$

(b) $E^2 \approx \Delta_L(r)^2$. Here we simply set $E^2 = \Delta_L(r)^2$ to estimate the contribution of the excited states. Up to numerical factors of order unity, we find

$$\rho_L(E;\mathbf{x})^{(n>0)} \sim \frac{N(0)\sqrt{v_F\,\Delta'(0)}}{\Delta_L(r)}\,(\Delta_L(r)^2/(2v_F\,\Delta'(0))^{5/2}. \tag{C.11}$$

APPENDIX D: THE RESULT OF CAROLI *ET AL.* [2] FOR THE LOW ENERGY DENSITY OF STATES/UNIT LENGTH

In this appendix we give a discussion of the result of Caroli *et al.* [2] for the low energy density of states/unit length of vortex and that obtained in Eq. (6.2) of the present work. As stated in Section 6, there appears to be a factor of two discrepancy between the two results; Caroli *et al.* claiming their result includes a single spin projection while the result given in Eq. (6.2) contains a contribution from both spin projections.

Our understanding is that in their work, Caroli *et al.* effectively evaluated Tr $\delta(E-H)$ in the limit $E \to 0_+$, where H is given by Eq. (2.2) of the present work and Tr denotes a trace in both matrix and configuration space. The matrix trace, in a particle hole symmetry approximation, is equivalent to the spin factor of 2 appearing in Eq. (2.3) and hence it is our opinion that their result does indeed include both spin projections. There does not seem any way to test the result of Caroli *et al.* by taking the normal limit $\Delta \to 0$, since they only present a spatially integrated quantity which will diverge in this limit. We, by contrast, have expressions for local quantities and are in the position to take the $\Delta \to 0$ limit to test that no spurious factors of 2 have been introduced by the methods used in this work. The only drawback of taking this limit is that no particular emphasis is given to the state of interest in this discussion, namely $n = 0$; the normal limit just tests to see that correct spectral weight is given to reasonably high energy states—as will be shown below.

We have taken the normal limit in two places of the present work: (a) in the Greens function of Eq. (3.14) which was then used in Eq. (2.3) and (b) in Eq. (5.6). In both cases the correct result of twice the free particle density of states/unit volume for a single spin projection, $2N(0)$, was obtained. We present details of the latter calculation in this appendix.

Normal Limit of the Local Density of States

To go to the normal limit of the local density of states $\rho_L(E;\mathbf{x})$, given in Eq. (5.6), we need to take the limit $\Delta'(0)\to 0$ at fixed E and r. From Eq. (4.4b), Ω is proportional to $\Delta'(0)$; thus we are effectively taking the limit $\Omega\to 0$. The typical excitation numbers $n\sim E^2/\Omega$ will, at fixed finite E, become very large and we can use the "semiclassical" or WKB limit of the wavefunctions:

$$\Phi_n(\zeta)^2 \approx \frac{\Omega}{\pi}\left(1\Big/\sqrt{\frac{2}{\mu}\left((n+1/2)\,\Omega-\mu\Omega^2\zeta^2/2\right)}\right) \tag{D.1}$$

$$\approx \frac{\Omega}{\pi}\frac{1}{\sqrt{2\Omega n/\mu}}. \tag{D.2}$$

Since very high excitation numbers are present we shall, furthermore, approximate the sum over n in Eq. (5.6) by an integral,

$$\rho_L(E;\mathbf{x}) \approx 2\pi v_F N(0)\int\frac{d\Omega_k}{4\pi}\sin\vartheta\left[\sum_s\int_0^\infty dn\,\frac{\Omega}{\pi}\frac{1}{\sqrt{2\Omega n/\mu}}\,\delta(E-s\sqrt{n\Omega})\right]. \tag{D.3}$$

On evaluation this leads to the correct result of $2N(0)$.

The reproduction of the correct free particle local density of states in the above calculation, albeit in a slightly non-trivial way, leads us to believe in the essential correctness of the methods used this work.

APPENDIX E: Incorporating the Vector Potential into the Phase of the Order Parameter

In this appendix we consider a contribution of the magnetic field that is important for states that extend a large radial distance from the vortex, for example, scattering states.

The contribution in question follows from replacing the momentum operator in Eq. (7.1) by the gauge covariant form that is present in Eq. (2.2a): $\mathbf{p}+e\mathbf{A}\sigma_3$. Projecting onto the $\hat{e}_1$ axis, as in Eq. (3.10a), leads to $p_1+e(\hat{e}_1\cdot\mathbf{A})\,\sigma_3$. We take $\mathbf{A}$ to point in the $\hat{\psi}\equiv(-\sin\psi,\cos\psi,0)$ direction (ψ is defined in Eq. (2.1)) and write $\mathbf{A}$ in the form

$$\mathbf{A}(\mathbf{r}) = \frac{a(r)}{2er}\hat{\psi} \tag{E.1}$$

with the function $a(r)$ having the properties [11]

$$\lim_{r\to 0} a(r)/r = 0 \tag{E.2}$$

$$\lim_{r\to\infty} a(r) = 1. \tag{E.3}$$

Equation (E.2) follows from the requirement of finiteness of the magnetic energy and Eq. (E.3) corresponds to the total magnetic flux of the vortex being one flux quantum, $\pi/e \equiv 2\pi\hbar/2e$ in dimensionful units. Furthermore, the function $a(r)$ is assumed to vary on the scale of the London magnetic penetration length λ_L.

A simple calculation indicates that

$$\hat{e}_1 \cdot \hat{\psi} = -\eta/r; \tag{E.4}$$

thus we can write

$$e(\hat{e}_1 \cdot \mathbf{A}) = -\eta \frac{a(r)}{2r^2} \equiv -\partial\chi(\zeta, \eta)/\partial\zeta \tag{E.5}$$

with

$$\chi(\zeta, \eta) = \int_0^{\zeta} d\zeta' \, \eta \frac{a(r')}{2r'^2}, \qquad r' = \sqrt{\zeta'^2 + \eta^2}. \tag{E.6}$$

We thus have

$$p_1 + e(\hat{e}_1 \cdot \mathbf{A})\,\sigma_3 = p_1 - \partial\chi(\zeta, \eta)\,\sigma_3/\partial\zeta \tag{E.7}$$

and we replace p_1 by the above quantity in Eq. (7.1), allowing us to write

$$\begin{aligned} &v_F \sin\vartheta(p_1 + e(\hat{e}_1 \cdot \mathbf{A})\,\sigma_3)\,\sigma_3 + \frac{\Delta(r)}{r}(\sigma_1\zeta + \sigma_2\eta) \\ &\quad = e^{i\chi\sigma_3}\left[v_F \sin\vartheta\, p_1\sigma_3 + \frac{\Delta(r)}{r}(\sigma_1\zeta + \sigma_2\eta)\, e^{2i\chi\sigma_3}\right] e^{-i\chi\sigma_3}. \end{aligned} \tag{E.8}$$

The virtue of Eq. (E.8) is that the original Hamiltonian, with the vector potential in the kinetic energy, has been related, via the unitary transformation $\exp[i\chi\sigma_3]$, to a new Hamiltonian, where the vector potential lies in the phase of the order parameter. It may be straightforwardly demonstrated that at large distances from the vortex, where $a(r) \to 1$ and $\Delta(r) \to \Delta_{BCS}$, the effective order parameter

$$\frac{\Delta(r)}{r}(\sigma_1\zeta + \sigma_2\eta)\, e^{2i\chi\sigma_3} \to \Delta_{BCS}\sigma_2 \,\mathrm{sign}(\eta) \tag{E.9}$$

with the right-hand side being equivalent, for all purposes, to the equilibrium order parameter of a uniform system.

The incorporation of the vector potential in the way outlined in this appendix is essential for scattering states. In the absence of the vector potential these states would be influenced at arbitrarily large distances from the vortex, whereas in reality the influence of the vortex is only of the order of the scale of variation of the function $a(r)$, i.e., the London magnetic penetration length, λ_L.

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