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Quantum dissipation of arbitrary strength from coupling to fermions

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Abstract. A single degree of freedom, analogous to the coordinate of a particle, is coupled to the density fluctuations of a filled sea of fermions. The fermions live in one spatial dimension and have a linear energy-momentum relation, as may be appropriate in the vicinity of the Fermi surface.

By using the equivalence of fermions and bosons in one space dimension, we show that the classical limit of the above theory leads to a Langevin equation for the coordinate of the 'particle' in which there are no constraints on the strength of the friction constant. This is in distinction to existing models of fermionic heat baths which have upper limits on the value of the friction constant.

The non-classical behaviour of the system is studied by determining the reduced equilibrium density matrix of the particle. The fermions, described by a functional integral over Grassmann fields, are uncoupled from the particle by a change of variables which is equivalent to a chiral gauge transformation. This generates a non-trivial Jacobian which is evaluated in an appendix and leads, in an appropriate limit, to identical results to those found by Caldeira and Leggett for an oscillator heat bath.

1. Introduction

In classical mechanics it is very common that a few key degrees of freedom (such as the coordinates of a particle) suffer dissipation of energy due to their coupling to many other degrees of freedom. In the recent past there has been a significant amount of work on the behaviour of these key degrees of freedom when the entire system (i.e. all degrees of freedom) are governed by quantum mechanics [1]. This has been largely devoted to understanding the behaviour of the magnetic flux in certain superconducting systems. The classical limit exhibited by these systems corresponds to one dimensional Langevin-like equations of motion.

The many degrees of freedom that are responsible for the dissipation in the classical limit were modelled, in the context of superconductivity, by Caldeira and Leggett as a heat bath consisting of harmonic oscillators†. Fairly soon after this a heat bath of fermions was considered by Chang and Chakravarty [3]. Their main conclusion was that there are great similarities between the effects of the two heat baths with one exception. The oscillator bath can have an arbitrarily large friction constant ($\equiv$ coupling constant) while the fermion bath has (for a finite number of fermion species)

† The literature on this subject is too extensive to be listed here. We simply note two review papers on the subject [1, 2].

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a maximum value of the friction constant. Recent work on the subject [4] has not changed this conclusion. It is the purpose of this paper to present a fermionic model of a heat bath that is capable of yielding arbitrary dissipation, while restricting itself to a single species of fermions.

2. Description of the model

We shall denote by q the single degree of freedom that, in the classical limit, exhibits dissipative dynamics and obeys a damped equation of motion. In what follows we shall refer to q as 'the coordinate of the particle' although, as we made clear in the introduction, its physical significance may be quite different from this. In the model under consideration we couple the coordinate to the density fluctuations of a filled sea of fermions. The fermions move in one space dimension, x, and have two-component field operators $\psi_a(x)$ $(a=1,2)$ which satisfy canonical anticommutation relations:

$$\{\psi_a(x), \psi_b(x')\} = \delta_{ab}\delta(x-x'). \tag{1}$$

The total Hamiltonian that describes the particle coupled to the heat bath is given by (summation over repeated indices is implicit):

$$H = H_\mathrm{b} + H_\mathrm{p} + H_\mathrm{int} + H_\mathrm{ct} \tag{2a}$$

where

$$H_\mathrm{b} = \int_{-\infty}^{\infty} \mathrm{d}x\, \psi_a^+(x)\sigma_{ab}^3 \frac{1}{\mathrm{i}}\frac{\partial}{\partial x}\psi_b(x) \tag{2b}$$

$$H_\mathrm{p} = \frac{p^2}{2m} + V(q) \tag{2c}$$

$$H_\mathrm{int} = q\int_{-\infty}^{\infty} \mathrm{d}x\, \Phi(x)\tfrac{1}{2}[\psi_a^+(x), \psi_a(x)] \tag{2d}$$

$$H_\mathrm{ct} = q^2\int_{-\infty}^{\infty} \mathrm{d}x \frac{1}{2\pi}\Phi^2(x). \tag{2e}$$

H_b is the Hamiltonian describing the dynamics of the uncoupled fermionic heat bath and σ^3 is the third Pauli spin matrix. This Hamiltonian can be viewed as a model of (one spin species of) electrons in the vicinity of the Fermi surface where their energy-momentum relation has been linearised. A quick way to see this is to note that the full energy-momentum relation is $\varepsilon = (k^2 - k_\mathrm{F}^2)/2m$ ($k_\mathrm{F} = mv_\mathrm{F}$ = Fermi momentum) and if $k = k_\mathrm{F} \pm \delta k$ then $\varepsilon \approx \pm v_\mathrm{F}\delta k$, and making the replacement $\delta k \to -\mathrm{i}\partial/\partial x$ leads, effectively to (2b). Thus the upper (lower) component of the field operator describes fermions moving to the right (left) with velocity $\approx v_\mathrm{F}$ $(-v_\mathrm{F})$. In the present work v_F is taken to be unity.

H_p is the standard result for the Hamiltonian of a particle moving in a potential $V(q)$.

H_int is the part of the total Hamiltonian representing the interaction between the particle and the heat bath. As has been extensively discussed and motivated in [1], we have taken this to be linear in the coordinate of the particle. There are two other factors in the interaction Hamiltonian.

One of these is the operator $\frac{1}{2}[\psi_a^+, \psi_a]$ which represents the density fluctuations of the Fermi field. The sense that $\frac{1}{2}[\psi_a^+, \psi_a]$ corresponds to the density fluctuations of the Fermi field is as follows. The deviation of the density from its mean value is $\psi_a^+\psi_a - \langle\psi_a^+\psi_a\rangle$. We can write for each bilinear combination of ψs, $\psi^+\psi = \frac{1}{2}[\psi^+, \psi] + \frac{1}{2}\{\psi^+, \psi\}$. The anticommutators cancel between the two terms in the density deviation (since both yield the state independent value $\delta(0)$). We are thus left with $\frac{1}{2}[\psi^+, \psi] - \langle\frac{1}{2}[\psi^+, \psi]\rangle$ for the density deviation. It is natural to take the expectation value of the commutator in an equilibrium state of the uncoupled Fermi bath. For the free fermion part of the total Hamiltonian of (2), H_b, this expectation value vanishes, leaving only the commutator for the density deviations.

The other factor appearing in the interaction Hamiltonian is the function $\Phi(x)$ which we describe as a 'form factor', which characterises the locality of the coupling of q to the Fermi field. In the present work we take $\Phi(x)$ to be a function that is strongly peaked about $x=0$. Physically, the interaction Hamiltonian describes a particle moving in a space orthogonal to that of the fermions and interacting with the fermions only in the vicinity of $x=0$.

With hindsight we have also included in the total Hamiltonian a 'counter term' H_{ct}, which has the effect of ensuring that $V(q)$ is the observed potential—i.e. the one appearing in the classical Langevin equation (see section 3).

3. Determining the equation of motion in the classical limit

Having written down the total Hamiltonian, it remains for us to demonstrate that in the classical limit it is capable of reproducing a Langevin equation for $q(t)$.

At first sight the procedure seems obvious: simply treat the variables appearing in the Hamiltonian (or Lagrangian) as classical quantities and work out their equations of motion. This procedure runs into difficulties when it is realised that the problem involves Fermi fields which, naively at least, do not appear to have a classical limit. There is a way out of this difficulty, however: namely to use the equivalence of fermions and bosons in one spatial dimension [5]. Once the theory is written in terms of bosons the original procedure may be used. To proceed, we write down, using standard principles [6], the Lagrangian equivalent to the Hamiltonian of (2)

$$L = \int dx[\bar{\psi} i\partial_t \psi + \bar{\psi}\sigma^3 i\partial_x \psi - q\Phi\bar{\psi}\psi] + \frac{m\dot{q}^2}{2} - V(q) - q^2 \int \frac{dx}{2\pi}\Phi^2. \qquad (3)$$

In this equation ψ and $\bar{\psi}$ are independent Grassmann fields and the time evolution kernal is a functional integral over ψ and $\bar{\psi}$ of $\exp(i\int L\,dt)$. By making the change of variables $\bar{\psi} \to \bar{\psi}\sigma^1$ and identifying $\gamma^0 = \sigma^1$, $\gamma^1 = -i\sigma^2$, $x^0 = t$, $x^1 = x$, we can make the fermion part of the Lagrangian look like that of a relativistic field theory

$$L = \int dx[\bar{\psi} i\gamma^\mu \partial_\mu \psi - q\Phi\bar{\psi}\gamma^0\psi] + \frac{m\dot{q}^2}{2} - V(q) - q^2 \int \frac{dx}{2\pi}\Phi^2. \qquad (4)$$

In this form the theory involving Fermi fields can be easily replaced by a theory having completely equivalent dynamics but now involving only boson fields. The replacement is implemented by the standard bosonisation rules [5]

$$\bar{\psi}\mathrm{i}\gamma^{\mu}\partial_{\mu}\psi \to \tfrac{1}{2}(\partial_{\mu}\theta)(\partial^{\mu}\theta) \equiv \tfrac{1}{2}(\partial\theta)^2$$
$$\bar{\psi}\gamma^0\psi \to -\frac{1}{\sqrt{\pi}}\partial_x\theta \tag{5}$$

where θ is the boson field replacing the fermions.

We thus obtain a Lagrangian with equivalent dynamics to the original Hamiltonian

$$L = \int \mathrm{d}x \left[\tfrac{1}{2}(\partial\theta)^2 + \frac{q\Phi}{\sqrt{\pi}}\partial_x\theta\right] + \frac{m\dot{q}^2}{2} - V(q) - q^2 \int \frac{\mathrm{d}x}{2\pi}\Phi^2. \tag{6}$$

We are now in a position to determine the classical content of the theory. The Euler–Lagrange equations that follow from the Lagrangian of (6) are

$$-\partial^2\theta - q\frac{\partial_x\Phi}{\sqrt{\pi}} = 0$$
$$-m\ddot{q} - V'(q) - q\int \frac{\mathrm{d}x}{\pi}\Phi^2 + \int \mathrm{d}x \frac{\Phi\partial_x\theta}{\sqrt{\pi}} = 0. \tag{7}$$

Eliminating θ from these (we have assumed that at $t=-\infty$, $\theta=0$, $\dot{\theta}=0$) leads to

$$m\ddot{q}(t) + V'(q(t))$$
$$= -\frac{1}{\pi}\int \mathrm{d}x\,\mathrm{d}x'\,\mathrm{d}t'\,\Phi(x)\partial_x^2 G(x-x', t-t')q(t')\Phi(x') - q(t)\int \frac{\mathrm{d}x}{\pi}\Phi^2 \tag{8}$$

where we have introduced the retarded Green function $G(x-x', t-t')$, which obeys

$$(\partial_t^2 - \partial_x^2)G(x-x', t-t') = \delta(x-x')\delta(t-t') \tag{9}$$

and is given by

$$G(x, t) = -\int \frac{\mathrm{d}\omega\,\mathrm{d}k}{(2\pi)^2}\frac{\mathrm{e}^{\mathrm{i}(kx-\omega t)}}{(\omega - k + \mathrm{i}0_+)(\omega + k + \mathrm{i}0_+)}. \tag{10}$$

Let us assume that the form factor $\Phi(x)$ is an even function whose Fourier transform is

$$\tilde{\Phi}(k) = \int \Phi(x)\,\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x. \tag{11}$$

The RHS of (8) may, using (10), (11), be written in the form

$$\int \frac{\mathrm{d}k}{2\pi^2}\int_{-\infty}^{\infty}\mathrm{d}t'\,\theta(t-t')[\tilde{\Phi}(k)]^2\left[\frac{\mathrm{d}}{\mathrm{d}t'}\cos k(t-t')\right]q(t') - q(t)\int\frac{\mathrm{d}x}{\pi}\Phi^2.$$

Integrating this by parts leads to

$$\text{RHS} = -\int_{-\infty}^{t}\mathrm{d}t'\,\gamma(t-t')\dot{q}(t') \tag{12}$$

with

$$\gamma(t) = \int \frac{\mathrm{d}k}{2\pi^2}[\tilde{\Phi}(k)]^2\cos kt \tag{13}$$

$$\equiv \int \frac{\mathrm{d}x}{\pi}\Phi(x)\Phi(x+t). \tag{14}$$

Thus (8) is

$$m\ddot{q}(t)+V'(q(t))=-\int_{-\infty}^{t}\mathrm{d}t'\,\gamma(t-t')\dot{q}(t'). \tag{15}$$

As in other studies of a heat bath interacting with a single degree of freedom, the coupling between them does not only produce dissipative terms in the equation of motion, but also additional potential terms [1, 2]. As we have seen in the steps leading to (15), the role of H_{ct} is to precisely cancel these additional terms, thereby ensuring that the potential appearing in H, $V(q)$, is the one appearing in the equation of motion.

Let us now consider a specific choice for the function $\Phi(x)$. It is convenient to specify its Fourier transform $\tilde{\Phi}(k)$; we take

$$\tilde{\Phi}(k)=\lambda\,\mathrm{e}^{-|k|/\omega_{\mathrm{c}}}\qquad \omega_{\mathrm{c}}=\text{constant}. \tag{16}$$

Then in the limit $\omega_{\mathrm{c}}\to\infty$, $\gamma(t-t')\to(\lambda^2/\pi)\delta(t-t')$ and a Langevin equation is obtained (if ω_{c} is held to be finite, the system has a memory with a timescale $\sim\omega_{\mathrm{c}}^{-1}$). Note that there are no constraints on the value of the parameter λ^2/π, which plays the role of a friction constant, hence we infer that despite having a fermion heat bath, the system can have an arbitrarily large friction constant.

4. Quantised theory at finite temperatures

To reinforce the last statement of section 3 we shall, in this section, show directly from the fermion model how to obtain the reduced density matrix for the particle, and hence make the connection with existing results.

The reduced density matrix for the particle at temperature $T=1/\beta$ is

$$\rho(q,q';\beta)=\mathrm{Tr}_b\langle q|\mathrm{e}^{-\beta H}|q'\rangle/Z \tag{17}$$

where Tr_b denotes a trace over the fermion degrees of freedom and $Z=\mathrm{Tr}_b\,\mathrm{e}^{-\beta H_b}$ is the partition function of the uncoupled Fermi bath. From this reduced description we can calculate all equilibrium quantities that relate to the particle alone. In functional integral representation, (17) has the form

$$\rho(q,q';\beta)=\int_{q',0}^{q,\beta}\mathrm{d}[q]F[q]\exp-\int_0^{\beta}\mathrm{d}t\left(\frac{m\dot{q}^2}{2}+V(q)\right) \tag{18}$$

where

$$F[q]=\frac{1}{Z}\int_{\mathscr{A}}\mathrm{d}[\bar{\psi}]\,\mathrm{d}[\psi]\exp-\int_0^{\beta}\mathrm{d}t\int\mathrm{d}x\left(\bar{\psi}(\partial_t-\mathrm{i}\sigma^3\partial_x+q\Phi)\psi+\frac{q^2\Phi^2}{2\pi}\right) \tag{19}$$

and $\mathscr{A}$ denotes evaluating the Grassmann integral over fields that are antiperiodic over β. In order to evaluate $F[q]$, we shall recast our theory in the form of a Euclidean field theory, we shall then be able to draw upon existing results in the literature. To this end we first make the change of variables $\bar{\psi}\to\bar{\psi}\sigma^1$, and furthermore define

$$\begin{array}{llll} x_4=t & x_1=x & \gamma_4=\sigma^1 & \gamma_1=-\sigma^2 \\ \gamma_5\equiv\mathrm{i}\gamma_4\gamma_1=\sigma^3 & \mathrm{i}eA_4=-q\Phi & \mathrm{i}eA_1=0. & \end{array} \tag{20}$$

(The γ matrices obey $\gamma_\mu\gamma_\nu=\delta_{\mu\nu}+\mathrm{i}\varepsilon_{\mu\nu}\gamma_5$, with $\varepsilon_{14}=-\varepsilon_{41}=1$, $\varepsilon_{11}=\varepsilon_{44}=0$.)

With (20) we can write

$$F[q]=\exp\left(-\int q^2\Phi^2\frac{\mathrm{d}x\,\mathrm{d}t}{2\pi}\right)\int_{\mathscr{A}}\mathrm{d}[\bar{\psi}]\,\mathrm{d}[\psi]\,\mathrm{e}^{-S}/Z \tag{21a}$$

where

$$S=\int_0^\beta \mathrm{d}t\int \mathrm{d}x\,\bar{\psi}\gamma_\mu(\partial_\mu-\mathrm{i}eA_\mu)\psi. \tag{21b}$$

We next perform a gauge transformation to make $\partial_\mu A_\mu=\partial_4 A_4+\partial_1 A_1=0$. We achieve this by making the change of variables

$$\psi\to\mathrm{e}^{\mathrm{i}\Lambda}\psi \qquad \bar{\psi}\to\bar{\psi}\,\mathrm{e}^{-\mathrm{i}\Lambda}. \tag{22a}$$

The transformed A that results from this is

$$\bar{A}_\mu=A_\mu-\partial_\mu\Lambda/e. \tag{22b}$$

Then $\partial_\mu\bar{A}_\mu=0$ implies

$$\Lambda=\frac{1}{\partial^2}\,e\partial_\mu A_\mu \tag{22c}$$

$(\partial^2=\partial_4^2+\partial_1^2)$.

Given the action S of (21b) with $\bar{A}$ obeying $\partial_\mu\bar{A}_\mu=0$, we can completely decouple the fermions from the $\bar{A}$ field by using the chiral transformation of Roskies and Schaposnik [7]

$$\psi\to\mathrm{e}^{\mathrm{i}\gamma_5\alpha}\psi \qquad \bar{\psi}\to\bar{\psi}\,\mathrm{e}^{\mathrm{i}\gamma_5\alpha} \tag{23a}$$

with α chosen to cancel the $\bar{A}$ dependence in S: $e\gamma_\mu\bar{A}_\mu=\gamma_\mu\partial_\mu(\gamma_5\alpha)$. This has the solution

$$\alpha=\mathrm{i}e\frac{1}{\partial^2}\partial_\alpha\varepsilon_{\alpha\beta}\bar{A}_\beta. \tag{23b}$$

A non-trivial consequence of the change of variables, (23), was discovered by Fujikawa, namely that the Jacobian of the transformation, J, is not unity [8]. After some calculation [7], it is found to be

$$J=\exp-\frac{1}{2\pi}\int \mathrm{d}x\int \mathrm{d}t\,\alpha\partial^2\alpha. \tag{24}$$

Putting together the preceding equations we see that in (21a) the functional integral divided by Z reduces simply to the Jacobian J. Thus

$$F[q]=\exp\left(-\int\frac{\mathrm{d}x\,\mathrm{d}t}{2\pi}q^2\Phi^2\right)J. \tag{25}$$

Details of the evaluation of $F[q]$ are contained in the appendix which obtains the result (in the limit $\omega_c\to\infty$)

$$F[q]=\exp\left[-\frac{\lambda^2}{8\beta^2}\int_0^\beta \mathrm{d}t\int_0^\beta \mathrm{d}t'\left(\frac{q(t)-q(t')}{\sinh[\pi(t-t')/\beta]}\right)^2\right]. \tag{26}$$

We may therefore write for the reduced density matrix of the particle

$$\rho(q,q';\beta)=\int_{q,0}^{q',\beta}\mathrm{d}[q]\exp(-S_{\mathrm{eff}}[q]) \tag{27}$$

where the effective action S_{eff} is given by

$$S_{\text{eff}}[q]=\int_0^\beta \mathrm{d}t\left(\frac{m\dot{q}^2}{2}+V(q)\right)+\frac{\lambda^2}{8\beta^2}\int_0^\beta \mathrm{d}t\int_0^\beta \mathrm{d}t'\left(\frac{q(t)-q(t')}{\sinh[\pi(t-t')/\beta]}\right)^2. \tag{28}$$

This result is equivalent to that found by Caldeira and Leggett for a bath of oscillators [1, 2]. We conclude that the Hamiltonian of (1), involving fermions, leads to identical behaviour to the model considered by Caldeira and Leggett. In particular, there is no limitation on the strength of the dissipation.

5. Discussion

The model we have considered in this paper involves a particle coupled linearly to the density fluctuations of fermions. By using methods fairly specific to fermions moving in one space dimension, we have shown explicitly that this model reproduces the Langevin equation in the classical limit and the equilibrium results of the (oscillator) model of Caldeira and Leggett [1, 2]. It should therefore be viewed as equivalent to their model—despite the apparently very different nature of the heat baths. We believe that the difference of the results of this paper compared with those of other authors who consider Fermi heat baths lies in the fact that we used a Fermi sea that was unbounded from below, whereas they had a finite energy bandwidth. Clarification of this point would be welcome.

We have, in section 2, already noted that the particle does not move in the same space as the fermions—we have no notion of the particle moving along x. Rather, the picture we have is of the particle moving orthogonally to the fermions. It is possible to consider models of a different kind in which the particle shares a common space with—moves along—the fermions. The coupling in this case might be of the density-density form $\int \mathrm{d}x\,\Phi(q(t)-x)\frac{1}{2}[\psi^+(x),\psi(x)]$ where, as previously, Φ is a function peaked about zero argument. Such a situation is easily accommodated in the framework described above, although detailed consideration of the form of H_{ct}, if it is required at all, is necessary in this case.

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Appendix

In this appendix we present the steps leading to the form of the functional $F[q]$ given in (25); $F[q]=J\exp[-(1/2\pi)\int \mathrm{d}x\,\mathrm{d}t\,q^2\Phi^2]$.

Equations (24) and (23b) yield

$$J=\exp\left(\frac{e^2}{2\pi}\int \mathrm{d}x\int \mathrm{d}t\,\varepsilon_{\alpha\beta}\partial_\alpha\bar{A}_\beta\frac{1}{\partial^2}(\varepsilon_{\alpha\delta}\partial_\gamma\bar{A}_\delta)\right). \tag{A1}$$

Replacing $\bar{A}$ by A (since $\varepsilon_{\alpha\beta}\partial_\alpha\partial_\beta\equiv 0$), using (20) and integrating by parts twice, we obtain

$$J = \exp \Omega$$

$$\Omega = \frac{1}{2\pi} \int \mathrm{d}x\, \mathrm{d}t\, \mathrm{d}x'\, \mathrm{d}t'\, q(t) q(t') \Phi(x) \Phi(x') \partial_x \partial_{x'} G(x, x'; t-t') \tag{A2}$$

where the Green function G is given by

$$-\frac{1}{\partial^2} = G(x, x'; t-t') \qquad (\partial^2 = \partial_x^2 + \partial_t^2). \tag{A3}$$

It is convenient at this point to specify the Green function $G(x, x'; t-t')$ of (A3). Going back to (22) of the main text, we note the following. The transformed Grassmann fields are chosen, like the original fields, to be antiperiodic functions of t over the interval β. This results in Λ, and hence G, being periodic functions of t over the same interval, β:

$$G(x, x'; t+\beta) = G(x, x'; t) \tag{A4}$$

In order to have a well defined spatial behaviour of the Green function, we impose Dirichlet boundary conditions at $-L$ and L:

$$G(-L, x'; t) = 0 = G(L, x'; t) \tag{A5}$$

It is a straightforward matter to determine G:

$$G(x, x'; t) = -\frac{1}{2\beta} \sum_{n=-\infty}^{\infty} \frac{\cosh \omega_n (x + x') - \cosh \omega_n (2L - |x - x'|)}{\omega_n \sinh 2\omega_n L}$$

$$\omega_n = \frac{2\pi n}{\beta}. \tag{A6}$$

The original Hamiltonian, (2), had the fermions moving over $(-\infty, \infty)$, and we shall take the limit $L \to \infty$ to recover this, although not in (A6), where the term with $n = 0$ diverges with L.

Proceeding with the evaluation of J, we note that the limit $L \to \infty$ can be straightforwardly taken in (A2), where $\partial_x \partial_{x'} G(x, x'; t-t')$ appears. The differentiations annihilate constant terms which diverge in the limit $L \to \infty$ and we can, in (A2), effectively replace the limiting form of $G(x, x'; t-t')$ by

$$G_\infty(x - x'; t - t') = \frac{1}{2\beta} \left[\sum_{n \neq 0} \left(\frac{1}{|\omega_n|} \mathrm{e}^{-|\omega_n||x-x'|} \mathrm{e}^{-\mathrm{i}\omega_n (t-t')} \right) - |x - x'| \right]. \tag{A7}$$

While G_∞ obeys the same equation as G, it does not have the same spatial boundary behaviour. Henceforth we shall replace G by G_∞, and thus have taken the $L \to \infty$ limit.

Continuing, we use the identities

$$q(t) q(t') = \tfrac{1}{2}[q^2(t) + q^2(t')] - \tfrac{1}{2}[q(t) - q(t')]^2 \tag{A8}$$

$$\begin{aligned} \partial_x \partial_{x'} G_\infty(x - x'; t - t') &= -\partial_x^2 G_\infty(x - x'; t - t') \\ &= \partial_t^2 G_\infty(x - x'; t - t') + \delta(x - x') \delta(t - t') \end{aligned} \tag{A9}$$

and write (A2) as

$$\Omega = \Omega_1 + \Omega_2 \tag{A10}$$

$$\Omega_1=\frac{1}{4\pi}\int \mathrm{d}x\,\mathrm{d}x'\,\mathrm{d}t\,\mathrm{d}t'\,[q^2(t)+q^2(t')]$$
$$\times[\partial_t^2 G_\infty(x-x';\,t-t')+\delta(x-x')\delta(t-t')]\Phi(x)\Phi(x') \tag{A11}$$

$$\Omega_2=-\frac{1}{4\pi}\int \mathrm{d}x\,\mathrm{d}x'\,\mathrm{d}t\,\mathrm{d}t'\,[q(t)-q(t')]^2\partial_t^2 G_\infty(x-x';\,t-t')\Phi(x)\Phi(x'). \tag{A12}$$

Ω_1 can be simplified by using the fact that $G(x-x';\,t-t')$ is even and periodic (over β) in $t-t'$. Thus

$$\Omega_1=\int_0^\beta \mathrm{d}t\,\tfrac{1}{2}q^2(t)\int \mathrm{d}x\,\mathrm{d}x'\int_0^\beta \mathrm{d}t'\,\frac{1}{\pi}$$
$$\times[\partial_{t'}^2 G_\infty(x-x';\,t')+\delta(x-x')\delta(t-t')]\Phi(x)\Phi(x'). \tag{A13}$$

The time derivative of the Green function integrates to zero, being itself periodic in β, thus

$$\Omega_1=\int_0^\beta \mathrm{d}t\,q^2(t)\int\frac{\mathrm{d}x}{2\pi}\Phi^2(x). \tag{A14}$$

This term exactly cancels with the contribution of H_{ct}, i.e. the factor multiplying J in (25).

Equation (A12) can, with the use of the Poisson summation formula and the explicit forms for G_∞ (A7) and Φ (11), (16), be expressed as

$$\Omega_2=-\int_0^\beta \mathrm{d}t\int_0^\beta \mathrm{d}t' K(t-t')[q(t)-q(t')]^2 \tag{A15}$$

where

$$K(t-t')=\frac{\lambda^2}{8\pi^2}\sum_n\frac{1}{(|t-t'+n\beta|+\omega_c^{-1})^2}. \tag{A16}$$

If, in (A16), we neglect ω_c^{-1}, then Ω_2 may be written as

$$\Omega_2=-\frac{\lambda^2}{8\beta^2}\int_0^\beta \mathrm{d}t\int_0^\beta \mathrm{d}t'\,\frac{[q(t)-q(t')]^2}{\sinh^2[\pi(t-t')/\beta]}. \tag{A17}$$

We thus find that $F[q]$ is given by

$$F[q]=\exp(\Omega_2[q]). \tag{A18}$$

If we periodically continue $q(t)$ such that $q(t+\beta)=q(t)$, then (A17) can be expressed in the same form as Caldeira and Leggett's result [1,2]:

$$\Omega_2=-\frac{\lambda^2}{8\pi^2}\int_{-\infty}^{\infty}\mathrm{d}t\int_{-\infty}^{\infty}\mathrm{d}t'\left(\frac{q(t)-q(t')}{t-t'}\right)^2. \tag{A19}$$

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