# On a Theory of Precise Neural Control in a Noisy System

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**Abstract** In this paper, we introduce a novel computational paradigm based on modern control and optimization theory and biological observations. We investigate the 'minimum-variance principle' of a controlled dynamical system with noise, assuming that the noise inherent to the control signal is sub-Poisson. In this case, we find that the optimal solution of the stochastic controller is not an explicit function but is composed of a parameterized measure. Moreover, in contrast to the supra-Poisson or Poisson noise, this sort of parameterized measure can achieve precise control performance even in the presence of noise.

## 1 Introduction

The purpose of this paper is to introduce a mathematical framework to realize precise neural control in a noisy system. The initial motivation of the paper comes from several biological observations. Noise is believed to be inevitable since it is an intrinsic component of the signal and furthermore its magnitude could also strongly depend on the signal magnitude [1]. However, as reported in [2], the movement error is believed mainly due to inaccuracies of the neural-sensor system,

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and not associated with the neural-motor system, which implies that the neural-motor system may be precisely controlled, even with randomness. A key feature of the neural signal is that it is locally distributed and likely to have only three states, namely inactive, excited, and inhibited. To make progress in understanding how precise movement control can be achieved in a noisy environment, we shall investigate theoretical relationships which may connect the observed activity of neurons with precise control performance.

In a mathematical form, the neural control problem can be expressed as minimizing the execution error caused by the noise inherent in the control signals [3]. One characteristic of the noise is the dispersion index, $\alpha$, which relates the variance in the control signal to the mean control signal and hence describes the statistical regularity of the control signal. When the variance in the control signal is proportional to the $2\alpha$-th power of the mean control signal the dispersion index of the control noise is said to be $\alpha$. It was shown in [1, 3] that an optimal solution of analytic form can be found when the stochastic control signal is supra-Poisson, i.e., when $\alpha \geq 0.5$. However, the resulting control is not precise and a non-zero execution error arises.

In the present work, thanks to an elegant theory developed by Young (Young measure) [4, 5], we introduce some of mathematical principles linking the regularity of the control signal noise and the precision of the resulting control performance. We consider two examples of neural control: saccadic eye movement control and straight-trajectory arm movement control, where neural spikes act as control signals, which are formulated as Gaussian processes with signal dependent variances. Our results show that if the control signal is less random than a Poisson process (i.e., $\alpha < 0.5$) then the control optimization problem naturally involves solutions with a specific character (parameterized measure optimal solutions), which can achieve precise control.

## 2 Methods/Models

The purpose of our control task is to minimize the variance of the final 'value' of a dynamical system under a constraint on its average activity. That is,

$$\begin{cases} \min_{\lambda(t)} \int_T^{T+R} \mathrm{var}[x(t)]dt, \\ \text{subject to} : \frac{dx}{dt} = a(x(t), t) + b(x(t), t)u(t) \\ x(0) = x_0;\ E[x(t)] = z,\ t \in [T, T+R]; \\ \lambda_i(t) \in [-M_Y, M_Y],\ t \in [0, T+R]. \end{cases} \tag{1}$$

Here, $\mathrm{var}(\cdot)$ and $E(\cdot)$ represent variance and expectation respectively, $x(t)$ is a state vector while $u(t) = [u_1(t), \cdots, u_m(t)]^\top$ is a controller vector, $a(x,t)$ denotes the uncontrolled dynamical system and $b(x,t)$ is the gain matrix with respect to $u$. Let $u_i(t) = \lambda_i(t) + \zeta_i(t)$, where $\lambda_i(t)$ denotes the mean control signal and each $\zeta_i(t)$ is an independent white noise with the properties $E(\zeta_i(t)) = 0$ and $E(\zeta_i(t)\zeta_j(t')) = \sigma_i(t)\sigma_j(t')\delta(t-t')\delta_{ij}$, while $\delta(\cdot)$ is a Dirac delta function and $\delta_{ij}$ a Kronecker delta. The noise fluctuation $\sigma_i(t)$ explicitly depend on the magnitude of the signal: $\sigma_i = \kappa_i|\lambda_i(t)|^\alpha$, with $\kappa_i > 0$, and $\alpha$ is the dispersion index of the control process. The aim of control is to let $x(t)$ reach a target $z$ at time $t = T$ and stay there for the period $[T, T+R]$.

Due to limited space, we cannot provide any details in the present paper, but give a summary of the main ideas. The mathematical contents can be found in our other papers. The abstract Hamiltonian minimum (maximum) principle (AHMP) [6] provides a necessary condition for the optimal solution, which is composed of the points that minimize the integrand function of the Hamiltonian (IFH). This principle indicates that the optimal solution should be a minimum of the given IFH for each $t$. If the control noise is supra-Poisson or Poisson, i.e., $\alpha \geq 0.5$, then the IFH is convex (or semi-convex), which implies that there is a unique minimum of the IFH for each $t$. Hence the optimal solution is an explicit function, in the sense that for each $t$, $\lambda(t)$ is the unique value that minimizes the IFH. If, however, the control signal is sub-Poisson, i.e., $\alpha < 0.5$, then no explicit function $\lambda(t)$ exists as the optimal solution, since the IFH is not convex. However, an optimal solution that is not an explicit function but a parameterized measure, $\{\eta_t(\cdot)\}$, exits. It is called 'Young measure' following [4, 5] and yields a set of values on which a measure (i.e., a weighting) $\eta_t(\cdot)$ is defined for each $t$. And, the optimal solution of Young measure has the form $\eta_t(\cdot) = \eta_{1,t}(\cdot) \times \cdots \eta_{m,t}(\cdot)$, with

$$\eta_{i,t}(ds) = [\mu_i(t)\delta(s - M_Y) + \nu_i(t)\delta(s + M_Y) + (1 - \mu_i(t) - \nu_i(t))\delta(s)]ds \tag{2}$$

with $\mu_i(t)$ and $\nu_i(t)$ non-negative and $\mu_i(t) + \nu_i(t) \leq 1$, $\mu_i(t)\nu_i(t) = 0$. In addition, we can derive that

$$\min_\eta \sqrt{\int_0^T \mathrm{var}[\phi(x,t))]dt} = O(1/(M_Y^{1/2-\alpha})), \tag{3}$$

as $M_Y \to \infty$. This implies the execution error approaches zero as $M_Y$ goes to infinity if $\alpha < 0.5$. This is in clear contrast to the situation where the control signals are Poisson or more random than Poisson (i.e., $\alpha \geq 0.5$) where the optimal control signal is an ordinary function, not a parameterized measure, and the variance in control performance cannot approach zero.

## 3 Results

We consider two examples of neural controls, where the control signal is described as a Gaussian process: $\lambda(t) + \sigma(t)dW_t/dt$, with the noise depending on the frequency $\lambda(t)$, that is $\sigma(t) = \kappa|\lambda(t)|^{\alpha}$ for some $\alpha > 0$, $\kappa > 0$. Then, the underlying dynamical system can be formulated as Itô diffusion.

First, we consider the model (4) of saccadic eyeball movements, which was studied in [7].

$$\ddot{x} = -\frac{1}{\tau_1\tau_2}x - \frac{\tau_1+\tau_2}{\tau_1\tau_2}\dot{x} + \gamma\Big[\lambda(t)+\kappa|\lambda(t)|^{\alpha}(t)dW_t/dt\Big],\ x(0)=0,\ \dot{x}(0)=0. \tag{4}$$

Here $x$ is the position of the eyeball, $\gamma$, $\tau_{1,2}$ are positive parameters of the oculomotor plant, and $\lambda(t) + \kappa|\lambda(t)|^{\alpha}dW_t/dt$ describes the control signal accompanying with signal-dependent noise [1]. The control object is to let $x(t)$ reach a target $D$ at time $t = T$ and stay there for a period $[t, T + R]$. We revisit this problem via the idea of Young measure. As shown in Fig. 1A(a–c) with $\alpha = 0.25$ ($< 0.5$), one can see that the control signal is localized (Fig. 1A(b)) and the performance of control is precise (Fig. 1A(b)), in comparison to the case $\alpha > 0.5$ which cannot achieve a precise performance (Fig. 1A(c)).

Second, we consider a more complicated model of the arm movement related to biological signal control. The sensorimotor transformations are often formalized in terms of coordinate transformation. The nonlinearity arises from the geometry of the joints. For simplicity, we neglect gravity and viscous forces, and formulate the model as (5),

$$N(\theta_1,\theta_2)\begin{bmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\end{bmatrix} + C(\theta_1,\theta_2,\dot{\theta}_1,\dot{\theta}_2)\begin{bmatrix}\dot{\theta}_1\\ \dot{\theta}_2\end{bmatrix} = \gamma_0\begin{bmatrix}Q_1\\ Q_2\end{bmatrix},\ \theta_1(0) = -\frac{\pi}{2}, \theta_2(0) = \frac{\pi}{2},$$

$$\dot{\theta}_{1,2}(0) = 0$$

$$\text{with } N = \begin{bmatrix} I_1 + m_1r_1^2 + m_2l_1^2 + I_2 + m_2r_2^2 + 2k\cos\theta_2 & I_2 + m_2r_2^2 + k\cos\theta_2 \\ I_2 + m_2r_2^2 + k\cos\theta_2 & I_2 + m_2r_2^2 \end{bmatrix},$$

$$C = k\sin\theta_2\begin{bmatrix}\dot{\theta}_2 & \dot{\theta}_1 + \dot{\theta}_2\\ \dot{\theta}_1 & 0\end{bmatrix},\ Q_i = \lambda_i(t) + \kappa_0|\lambda_i(t)|^{\alpha}dW_1/dt, \tag{5}$$

where $\theta_{1,2}$ are the angles between upper arm and horizontal direction, forearm and upper arm, respectively, $\lambda_{1,2}(t)$ are control signals to two directions accompanying with signal-dependent noises, and all other symbols ($m_{1,2}$, $I_{1,2}$, $r_{1,2}$ and $\kappa_0$) are constant parameters. The relation between the position of hand $(x(t), y(t))$ and the angles $\theta_{1,2}$ is $\theta_1 = \arctan(y(t)/x(t)) - \arctan(l_2\sin\theta_2/(l_1 + l_2\cos\theta_2))$ and $\theta_2 = \arccos[(x^2+y^2-l_1^2-l_2^2)/(2l_1l_2)]$. For the details of the model, please refer to [8]. We are to control the final hand position to reach the given target $H = [H_1, H_2]$. We can use a numerical approach to calculate an approximate solution, as shown in Fig. 1B(b). As it is shown in Fig. 1B(a), when $\alpha < 0.5$, the optimal localized solution has a precise control performance, in comparison to the case $\alpha > 0.5$,

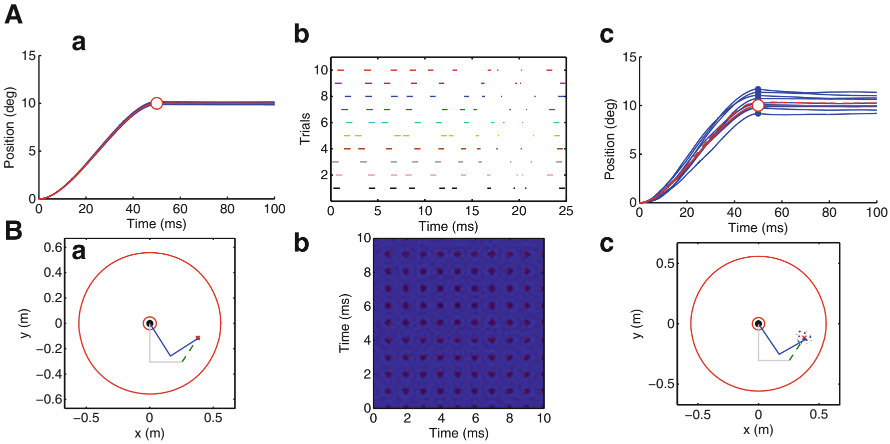


**Fig. 1** Optimal control and performance. The ODE is numerically solved by the Euler method with a time step 0.01 ms. Panel **A**: Saccadic eye movement model with parameters $\tau_1 = 224$ ms, $\tau_2 = 13$ ms, $\gamma = 1e-2$, $\kappa = 0.58$, $T = 50$ ms, $R = 50$ ms, $D = 10$ degree and $M_Y = 500$. (*a*) the dynamics of the position (in degree) under optimal control with $\alpha = 0.25$; the curves are plotted with ten overlaps (*blue lines*) by randomly picked initial values, the *red line* represents the mean over ten overlaps and the *red circle* is the pre-given position of the eye. (*b*) The localized sampling distributions of the value $M_Y$ which is picked by the Young measure $\eta_t(\cdot)$ with ten overlaps (indicated by different colors). (*c*) the dynamics of the position (in degree) under the optimal control with $\alpha = 1$; the curves are plotted with ten overlaps (*blue lines*) by randomly picked initial values and the *red line* represents the mean over ten overlaps. Panel **B**: Straight-trajectory arm movement model with parameters $m_1 = 2.28$ kg, $m_2 = 1.31$ kg, $l_1 = 0.305$ m, $l_2 = 0.254$ m, $I_1 = 0.022$ kg·m$^2$, $I_2 = 0.0077$ kg·m$^2$, $r_1 = 0.133$ m, $r_2 = 0.109$ m, $T = 650$ ms, $R = 10$ ms, $\phi = 3\pi/4$ and $M_Y = 20{,}000$. (*a*) the movement of the arm in a platform under the optimal control with $\alpha = 0.25$. The *red dash circle* represents error region over ten overlaps and the *gray line* is the theoretical trajectory. (*b*) the local distribution of the optimal Young measure, where x and y axes represent the $\eta_{1,2}$ respectively, and the *red points* represent that $\eta_{1,2}$ are picked values at $M_Y$ and otherwise in *dark blue*. (*c*) the movement of the arm in a platform under the optimal control with $\alpha = 1$ and the *red dash circle* represents error region over ten overlaps

which possess a deterministic solution but an unprecise performance as shown in Fig. 1B(c). The movement error also depends strongly on $\alpha$ and $M_Y$. The error decreases as $M_Y$ increases and the logarithm of the standard deviation is linearly dependent on the logarithm of $M_Y$ with a slope very near $\alpha$. This relation can be described as Eq. (3) but is not shown in this paper.

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